JUNE 2019

ΣNIGMATHIC

Produced by the AP Calculus Class
of
Manhattan High School for Girls

ΣNiGMATHiC

A Math Book
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June 2019

Message from the Principal

"Geometry," by former U.S. Poet Laureate Rita Dove, shares:

"I prove a theorem and the house expands:
The windows jerk free to hover near the ceiling,
The ceiling floats away with a sigh."

There is a beautiful relationship between Math and Poetry. Both languages make meaning by carefully placing symbols in just the right place. Both the number and the word can be manipulated to transform the meaning. Add or remove a number or a word, and you easily create either clarity or chaos or even a completely new perspective.

Our Math department approaches Mathematics as a language of expression. To that end, we focus on process and the diverse approaches to deriving at meaning. I am so grateful to our Math Chair, Mrs. Goldie Feinberg, for leading the department with strength and scholarship. Thank you to our Math educators: Mrs. Feinberg, Mrs. Manies and Mrs. Deitsch for teaching our girls how to think and express in the poetic language of Math.

Estee Friedman Stefansky

Mrs. Estee Friedman-Stefansky
Principal, General Studies

Foreword

"In mathematics, the art of proposing a question must be held of higher value than solving it." With these words Georg Cantor, the German mathematician who discovered set theory, expressed the notion that the questions we ask and the process it takes to finding the answer is of more value than the actual solution. Often in mathematics, the process of finding an answer to a question leads to entirely new branches of study. We discover patterns, applications and ideas that we never dreamed existed simply by setting out to finding a solution to a posed question. Even after searching for a solution, mathematical enigmas are often elusive and we may not have a clear-cut solution, but our observations and analyses allow us to appreciate how deep and pervasive mathematical concepts are embedded in the world Hashem created.

For this compilation, each student selected one of two topics; a known math problem where the solution took millennia to discover, or a currently evolving mathematical application. My hope is that the beauty, depth, and enigma of mathematics will continuously foster within our students a thirst for exploring the world's secrets.

Mrs. Goldie Feinberg
Chair, Math Department

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Map Out the Colors

Esther Guelfguat

Maps have always been necessary and a part of humanity. What many people don't know is how difficult it is to make a map. For centuries, scientists have struggled with what is the minimum number of colors that a map will need. Ever since the mid-1800's, mathematicians have tried to prove that a map can be drawn with only four colors, also known as the Four Color Theorem.

The quest to solve the Four Color Theorem began with Francis Guthrie (1831- 99), a student in London. In 1852, while he was coloring the regions on a map of England, he realized that he could color the map using only four colors in such a way that no two neighboring countries were the same color. Interested in this phenomenon, Guthrie wondered if this is true of all maps and if it could be proven mathematically. Therefore, he told his brother Frederick, a student at the University College in London, what he had hypothesized. Frederick was unable to solve the mystery so he asked his teacher, Augustus DeMorgan (1806-1871), a famous mathematician in the 1800's. DeMorgan was very impressed with the idea so he told it to his friend and colleague Sir William Rowan Hamilton (1805-65) (Walters 1).

After Hamilton received the Four Color Theorem from DeMorgan, all search for its authenticity went underground. This was true until Arthur Cayley (1821-95), English mathematician and lawyer, asked about the advances in the Four Color Theorem in the mathematical section of the Royal Society in July of 1878. Cayley asked if anyone had come up with a proof yet. This inquiry sparked the interest of mathematicians and they too began to look into the matter.

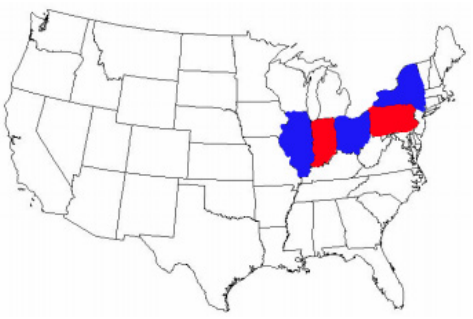

Figure 1
http://historyofmathematics.org/wp-content/uploads/213/09/2004-Walters.pdf

The first to come up with a proof was Sir Alfred Bray Kempe (1849-1922), a London lawyer and mathematician. His proof was published in the *American Journal*

of Mathematics about a year after Cayley's inquiry. Kempe was unaware at the time that his proof would become one of the most famous mathematical proofs. The theorem mainly uses graph theory, but other mathematical topics are applicable as well (Walters 3). Kempe developed a chain "of vertices that are colored with two alternating colors," which became the basis for his Four Color Theory. For example, in figure 1, there is a chain of blue and red colored states. In this case, California cannot be part of the chain because it does not have a border with the already colored states (Walters 5).

However, soon Kempe's proof was disproven. In 1890, Percy John Heawood (1861-1955) found an error in Kempe's theory but was not able to provide a different proof for the Four Color Theorem. Nevertheless, he acknowledged that Kempe's work did prove the Five Color Theorem and prompted more research on the topic.

By the 20th century, the Four Color Theorem shifted from an issue that British mathematicians were busy with to something American mathematicians were interested in (Walters 6). Oswald Veblen (1880-1960), a famous Princeton geometer, and his two students, George David Birkhoff (1884-1944) of Harvard and Philip Franklin (1898-1965) made major contributions to proving the Four Color Theorem (Walters 7).

Meanwhile, in 1936, Heinrich Heesch (1906-1995), a mathematician from the University of Hanover, started to work on the Four Color Theorem (Walters 8). He convinced a young student, Wolfgang Haken, from the University of Kiel in Germany who had attended one of Heesch's lectures, to attempt at solving the Four Color Theorem. Haken had been trying to solve three mathematical quandaries. The third on his list was the Four Color Theorem. After working on the first two, he finally decided to start the third. He contacted Heesch and they started to work together. Eventually, funding became a problem and Haken started to give up on solving the problem. At one conference he said that without the help of a computer the problem will remain unsolved (Walters 9). Luckily, Kenneth Appel, a mathematician and computer programmer at the University of Illinois, persuaded Haken that together they will finally solve the Four Color Theorem.

Finally, in the summer of 1976, Appel and Haken solved the Four Color Theorem. It took them several hundreds of pages of meticulous details and 1200 hours of "output on a powerful computer" for the theorem to be solved. They presented their proof at a conference in Toronto and later perfected it and published their proof (Walters 10). Many mathematicians

did not accept their proof, although it is mathematically correct, because it is almost entirely based on a computer's assistance (Weisstein). Therefore there have been other attempts at trying to prove the Four Color Theorem. Most notably, in 1996, Neil Robertson, Daniel Sanders, Paul Seymour, and Robin Thomas' came up with a proof which is considered the most "efficient" proof for this 150 year old problem (Walters 11).

Although making a map may seem like a simple endeavor, as is evident, it is not an easy task. For 150 years mathematicians struggled to find the minimum number of colors a map can have. Finally, this debate was brought to a close after the extensive efforts of Appel and Haken.

Appel and Haken working on their proof to the Four Color Theorem.
(https://las.illinois.edu/news/2017-10-18/celebrating-four-color-theorem)

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In Its Prime

Chayala Hauptman

The elementary definition of a prime number is any number that has factors of itself and 1. It is a number that cannot be factored. A fun mnemonic device to remember the first seven prime numbers is "In the early morning, astronomers spiritualized non-mathematicians." The number of letters in each word of this phrase, corresponds to each of the first seven prime numbers. But what are prime numbers and how do they enrich our math? Why are they so important to learn and why do they become a basic concept in any math class?

There are several valuable uses for prime numbers. Any number can be broken down to a product of primes. For example, 222= (2 x 3 x 37) or even a more extreme number like, 123,228,940= (2 x 2 x 5 x 23 x 79 x 3391). This is referred to as the Fundamental Theorem of Arithmetic. While this is interesting, there is a catch: it is impossible to apply the same algorithm of finding the prime factors of a smaller number compared to a larger one. The larger the number, even the most advanced supercomputer could take years to figure out the prime factors.

With this knowledge came a breakthrough in cyber security with the development of encryptions. Encryptions use an extremely large number as a key. Keys are made up of prime numbers, which tend to be extremely large. As previously mentioned, to break the key, one would need to use the Fundamental Theorem of Arithmetic and find the prime factors. The larger the number, the more impossible it becomes. Even artificial intelligence lacks the algorithmic knowledge to break this key.

An additional application of prime numbers is twin primes. This interesting discovery was coined by a German mathematician from the 19th century, Paul Stäckel. Twin primes are two prime numbers that have a gap of 2 between them (p, p+2). There are an infinite amount of twin numbers, but the proof of this fact is one of the most elusive concepts in math theory. Math enigmas, such as this, are the passion projects of many mathematicians, bent on solving math's greatest challenges.

In addition to simple math applications and cybersecurity, there is an interesting biological phenomenon that occurs according to prime numbers: the emergence of the cicadas from their underground habitats. A previous

theory explaining this absurdity was that prime numbers rarely overlap, so the cicadas, who come out during varying prime number intervals, will overlap rarely. For example, one that appears every 17 years and one that appears every 13 years will only overlap in 221 years. If they would emerge at the same time, it would cause a problem in getting food for all of the cicadas, but through their varied cycles, there is enough sustenance for everyone. An additional theory is that cicadas, who have a longer lifespan as they only leave their habitats rarely, tend to outlive the lifespans of their predators, therefore ensuring their security. How did such a pattern emerge? Through Darwinism: the cicadas that emerged in cycles closer to their predators were eaten and therefore did not perpetuate their species; the cicadas that exit on a prime-number basis have higher levels of survival, thus creating more like them.

Primes will most probably remain an enigma, but it will continue to be a part of our math system, a prime example that some things need to be taken on faith.

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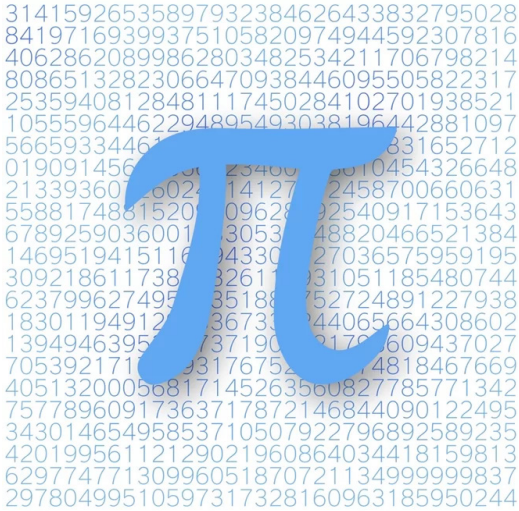
Easy as Pi

Rivka Sabel



Pi is defined as the ratio of a circle's circumference to its diameter. Aristotle hypothesized that Pi is an irrational number and that there does not exist a common unit of measure to calculate the circumference and radius of a circle. This was only proved two millennia later by Adrien-Marie Legendre (1752-1833). Pi is also considered a transcendental number, which means that there is no "root finding formula" that can be used to calculate Pi using rational numbers.

Calculating Pi has stunned mathematicians for many centuries. Ludolf Van Ceulen (1540-1610) calculated Pi to 35 places. By 1719 Thomas Fantet de Lagny calculated Pi to 127 digits. With the invention of computers came enormous progress of calculating Pi. First, between 1949- 1967, 500,000 digits were calculated by the ENIAC computer. Then in 2018, Peter Trueb used a computer to calculate Pi to 22,459,157,718,361 digits in 105 days! The world record is now held by Emma Haruka Iwao who took 125 days and used 25 virtual machines to calculate Pi to 31 trillion digits! But without a computer, Pi can be calculated using practical household items. Take a circular object such as a pizza cut into 8 slices, then calculate the line that would make the pizza slice an isosceles triangle. By adding up all the sides you will get a rough approximation of Pi and the more slices, the more accurate the approximation will be.

Aside from the amusing features of pie such that 3.14 backwards spells out pie, Pi has fascinating qualities to it. Pi is irrational and transcendental which means it will continue on forever without repeating or patterns. But many interesting sequences of numbers do appear in Pi's infinite numbers.

In the first million digits of π the sequence 12345 appears 8 times! The first six digits of π, 314159, appears in this order six times amongst the first ten million decimals of π. Any combination of numbers can be found in π, so someone's birthday and social security number can surely be found somewhere in Pi's infinite digits.

It is known that the world functions on mathematical principles, but Pi seems to be a magical number and appears in surprising places even in nature. In fact rivers bend to Pi. A river's sinuosity is the length of the river divided by the distance from its source. Interestingly, the average river has a sinuosity around 3.14.

The earth's gravity is even related to Pi! Earth's gravity is 9.8 m/s^2 which is very close to π ^2 which is equal to 9.87.

Pi has also been used in many other interesting and creative ways. There is even now a Pi language known as Pilish. Pilish is a dialect where the number of letters in successive words match the digits of Pi. Mike Keith even wrote an entire book in Pilish, called "Not a Wake."

But what is unknown to many is that Shlomo Hamelech knew Pi to 4 decimal places! Many believed that Shlomo thought the value of Pi to be 3, but the Vilna Gaon discovered that Shlomo's approximation of Pi was incredibly accurate for his ancient time. The passuk describes Shlomo's construction of a water basin in the *Beis Hamikdash* as:

"וַיַּעַשׂ אֶת־הַיָּם מוּצָק עֶשֶׂר בָּאַמָּה מִשְׂפָתוֹ עַד־שְׂפָתוֹ עָגֹל | סָבִיב וְחָמֵשׁ בָּאַמָּה קוֹמָתוֹ
וְקָו (כתיב וְקָוה) שְׁלֹשִׁים בָּאַמָּה יָסֹב אֹתוֹ סָבִיב:"

And he made the molten sea, ten cubits from brim to brim; it (was) round all about, and the height thereof (was) five cubits; and a line of thirty cubits did compass it round about. (Kings: 7:23)

This structure had a circumference of 30 cubits and a diameter of 10 cubits, which would make the approximation of π 30/10 or π=3. But the Vilna Gaon noticed that the Hebrew word for "line measure" was written as וְקָוה is this passuk, but written as וְקָו in *Divrei Hayamim*. The Vilna Gaon analyzed the *gematria* and found that וְקָוה has a *gematria* of 111 and וְקָו a 106. Taking the ratio of these two numbers 111/106 you get 1.0472 which is considered the "correction factor". After you multiply the correction factor by the value of π you get 3.1416 which is π correct to four decimal places!

Pi is considered amongst one of the most important numbers in math but is still underappreciated. It was discovered hundreds of years ago and still continues to fascinate mathematicians today who try to calculate as many decimal places of Pi possible. It also enthrals those who find Pi in astonishing places and subjects that no one would expect. Pi is infinite and it is safe to assume that just like we can never know all of Pi's digits, we can never be confident that we know all of where Pi is hidden in the world and all of its beauty.

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We'll Cross That Bridge

Ayelet Wein

The city of Königsberg was founded in 1254 and soon became a very important city and trading center, as it was strategically positioned on the river. Artwork from the Eighteenth Century shows how much Königsberg thrived and offered a comfortable lifestyle, due to the large amount of trade. The healthy economy allowed the people of the city to build seven bridges across the river which divided the city into four distinct regions.

The seven bridges were called Blacksmith's Bridge, Connecting Bridge, Green Bridge, Merchant's Bridge, Wooden Bridge, High Bridge, and Honey Bridge.

It has been told that the people of Königsberg would spend Sunday afternoons walking around their city. While walking, the people of the city would play a game and try to find a way to walk around the city, crossing each of the seven bridges only once. Even though none of the people of Königsberg could come up with a way to cross each of the bridges only once, they could not prove that it was impossible. Lucky for them, Königsberg was right near the town of St. Petersburg, where the famous mathematician, Leonhard Euler lived.

In his paper, 'Solutio problematis ad geometriam situs pertinentis', published in 1741, Euler explained both the solution to this specific problem, and any problem including any number of landmasses and any number of bridges.

First, Euler stated the general question he is trying to solve, and then started to explore different methods of finding a solution. Euler gave uppercase letters to represent each landmass and lowercase to represent each bridge. To represent the crossing of a bridge that began in landmass A

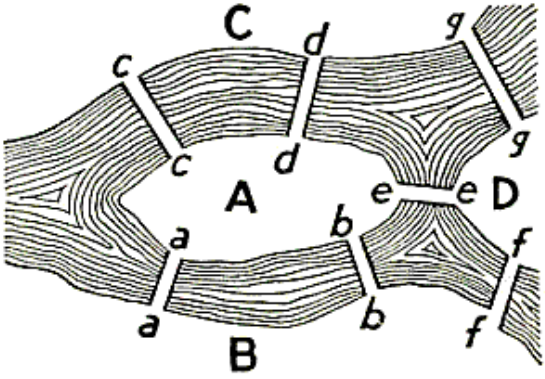

FIGURE 98. *Geographic Map: The Königsberg Bridges.*

and ended in B, Euler named it, AB. And if after traveling from landmass A to B, someone decides to move to landmass D, this would written as, ABD. Then Euler said that in ABDC, three bridges were crossed, even though there were four letters. To represent each crossing, there will be one more letter than bridges. Because of this, to solve the Königsberg Bridge Problem, with seven bridges, eight capital letters were necessary.

So, in order to solve the problem, Euler needed to come up with an eight-letter sequence or prove that it doesn't exist.

Before Euler did this for the Königsberg Bridge problem, he decided to find a rule to discover if there is a path for a more general problem. Euler stated that if there is an odd number of bridges connecting landmass A to other landmasses, add one to the number of bridges, and divide it by two, to find out how many total times A must be used in the path, where each bridge is used once and only once.

Once Euler came up with this rule, he then solved the Königsberg Bridge Problem. Since in Königsberg, there are five bridges that lead to A, it must be used in the path three times. Similarly, B, C, and D must appear twice since they all have three bridges that lead to them. Therefore 3(for A) + 2(for B) + 2(for C) + 2(for D) = 9. However, Euler previously stated, for seven bridges, there should only be an eight letter sequence. Eight and nine are not the same, proving that it is impossible to travel the seven Königsberg bridges only once.

After solving the Konigsberg problem, Euler continued to prove problems involving more general situations. He came up with a formula to solve any problem like this, no matter how many bridges or landmasses there were. If the sum of the number of times each letter must appear is one more then the total number of bridges, a journey can be made. However, just like the Königsberg problem, if the sum of the number of times each letter must appear is greater than one more than the number of bridges, a journey cannot be made.

With that, Euler proved how someone can solve more general forms of the problem, with three rules. First, if there are more than two landmasses with an odd number of bridges, it is not possible to travel each bridge only once. Second, if the number of bridges is odd for exactly two landmasses, then it is possible if it starts in one of the two odd numbered landmasses. Finally, Euler stated that if there are no regions with an odd number of landmasses, then the journey can be accomplished starting in any region.

Euler concludes that after one figures out that a path indeed exists, they must still go through the effort of writing out a path that works.

After Euler showed the steps to prove the Konigsberg problem, a somewhat simple work of mathematics came into play. What Euler did, which was new and creative, was viewing the Königsberg bridge problem abstractly, by using lines and letters to represent the landmasses and bridges. Graph theory is where a graph is a collection of vertices and edges. Because of this problem, a path in a graph, which contains each edge of the graph once and only once, is called an Eulerian path. Since Euler solved this famous problem, graph theory has become an important branch of mathematics. This problem is so important that it is mentioned in the first chapter of every Graph Theory book.

Graph theory is applied all the time to networks. Networks are a group of two or more devices that can communicate. They allow computers and individuals to communicate. For example, email and instant messaging. A graph can be made to represent electric circuits. What seemed like a simple problem, The Königsberg Bridge Problem, ultimately let to graph theory and networks, which are so integral to lives today.

In 1875, the people of Königsberg decided to build a new bridge, between landmass B and C, increasing the number of links of these two landmasses to four. This meant that only two landmasses had an odd number of links, which gave a solution to the problem. It is unknown if the creation of the extra bridge may have been subconsciously caused by the desire for a path that solves the town's famous problem.

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